Neutrosophic Modules

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Abstract

The objective of this thesis is to study neutrosophic $R$– module. Some basic features and definitions of the classical $R$– module are expanded. It is clear that every neutrosophic $R$– module over a neutrosophic ring is a $R$– module. Also, it is shown that an element of a neutrosophic $R$– module over a neutrosophic ring can be infinitely conveyed as a linear combination of some elements of the neutrosophic $R$– module. Neutrosophic quotient $R$– module and neutrosophic $R$– module homomorphism are also covered.

Keywords: Neutrosophic Ring; Neutrosophic Vector Space; Left Neutrosophic $R$– module; Right Neutrosophic $R$– module; Neutrosophic $R$– module Homomorphism

Introduction

In our life there is three kinds of logic. the first is classical logic which is gives the form “true or false, 0 or 1” to the values. The second is fuzzy logic was first advanced by Dr. Lotfi Zadeh in 1965s. It recognize more than true and false values which are considered simple. With fuzzy logic, propositions can be represented with degrees of truth and falseness. And the third is neutrosophic logic that is an extending fuzzy logic which includes indeterminacy $I$.

Since we live in a world filled in indeterminacy, the Neutrosophic found their method into modern research. We can introduce the Neutrosophic Measure and as a result the Neutrosophic Integral and Neutrosophic Probability in several methods, because there are different kinds of indeterminacies, depending on the problem and issue we have to fix. Indeterminacy is distinguished from randomness. Indeterminacy can be caused by physical space materials and type of construction, by items involved in the space, or by other factors. Space objects and structures are the main causes of indeterminacy and other factors can be considerable.

Florentin Smarandanche defined the idea of neutrosophy as a new type of philosophy in 1980. After he found the approach of neutrosophic logic and neutrosophic set where we have a percentage of truth in a subset $T$ and the same of falsity of the subset $F$, and a percentage of indeterminacy in the subset $I$ for every structure in neutrosophic logic where $T, I, F$ are subset of $[0,1]$. Therefore this neutrosophic logic is called an extension of fuzzy logic especially to intuitionistic fuzzy logic.

For more explanation, we can give this simple example: if we say “the weather is hot today”. In the classical logic we will say “yes or not, true or false”. However, in fuzzy logic we can say “it is 70% and 30% cold”. On the other hand, in neutrosophic logic we can say “It is 60-70% hot, 25-35% cold, and 10% indeterminate”.

In deed neutrosophic sets is the extension of classical sets, neutrosophic groups, neutrosophic ring, neutrosophic fields, neutrosophic vector spaces … etc. In the same way neutrosophic $R$– module is the generalization of classical $R$– module.

Using the idea of neutrosophic logic, Vasantha Kanadasamy and Florentin Smarandanches studied neutrosophic algebraic structures by using an indeterminate element $I$ in the algebraic structure and then combine $(I)$ with each element of the structure with respect to corresponding binary operation.

The indeterminate element $I$ is in order that if $*$ is ordinary multiplication the multiplication of many $(I)$ is $(I)$ itself and the inverse $I^{-1}$ is not defined and hence is not found. If we have $*$ as an ordinary addition, then the addition of many $(I)$ is $(I)$ itself. They call it neutrosophic element and the generated algebraic structure, is then termed as neutrosophic algebraic structure.
In 1995, Florentin Smarandache introduced the “neutrosophic set theory” to process the indeterminate and inconsistent information found generally in real cases.

In 2015 Salama introduced the concept of Neutrosophic Crisp set Theory to portray any event by a triple crisp structure. Moreover the work of Salama et al. formed a starting point to construct new types of neutrosophic mathematics and computer sciences. Hence, Neutrosophic set theory turned out to be a generalization of both the classical and fuzzy counterparts.

Neutrosophic logic has a broad applications in science, medicine, economics, chemistry, law etc. Therefore, neutrosophic structures are very significant and a broad area of study.

In addition to the introduction, this thesis contains in its second chapter suitable definitions and revision. Moreover, in the third chapter, the modules have been defined through the neutrosophic logic, some examples, theories, and proofs of its legitimacy and validity have been mentioned. Finally, in the fourth chapter, the homomorphisms has been studied with mentioning the theories and examples that support and confirm them.

**Fuzzy Set**

If we have \( U \) as an initial universe and if we take \( FU \) as a non-empty set in \( U \). A fuzzy set \( FU \) is defined as a set of arranged pairs \( \{(f, \mu_{FU}(f)) \} \) where \( f \in U \), the membership function \( \mu_{FU}: U \to [0,1] \) of \( FU \) and \( \mu_{FU}(f) \in [0,1] \) is the degree of membership function of element \( f \) in fuzzy set \( FU \) for any \( f \in U \) [1].

**Example:** Consider the universe of discourse \( U = \{3,5,6,7,9,10\} \). Then a fuzzy set holding the idea of 'large number' can be explained as \( A = \{(3,0),(5,0.1),(6,0.2),(7,0.3),(9,0.8),(10,1)\} \). With the considered universe, the numbers 3 is not 'large numbers', so the membership degree equals 0. Numbers 5–9 partially belong to the idea 'large number' with a membership degree of 0.1, 0.2, 0.3, 0.5 and 0.8, then number 10 is the largest number with a full membership degree.

**Neutrosophic Numbers**

The neutrosophic numbers formed as \( x(I) = y + zI \) or \( zI \) and \( y \) is defined as the determinate part on \( x(I) \) and \( zI \) is defined as the indeterminate part of \( x(I) \), with \( zI = (z, z)I \) . If both \( y, z \) are real numbers, then \( x(I) = y + zI \) is defined as a neutrosophic real number. If \( y, z \) or both are complex numbers, then \( x(I) = y + zI \) is defined as a neutrosophic complex number [2].

Example: Let \( x(I) = 5 + \sqrt{2}I \) be a real neutrosophic number which has 5 at the determinate part on \( x(I) \) and \( \sqrt{2}I \) the indeterminate part on \( x(I) \).

Similarly let \( z(I) = \sqrt{-1}I \) be a complex neutrosophic number which has 0 as the determinate part on \( z(I) \) and \( \sqrt{-1}I \) the indeterminate part on \( z(I) \).

**Neutrosophic Sets**

If we have \( U \) as an initial universe and if we take \( PH \) as a subset of \( U \). \( PH \) is defined as a neutrosophic set if it was an element \( p \in U \) goes back to \( PH \) in the following form:

\[
p = (T_{PH}(p), I_{PH}(p), F_{PH}(p)) \in U
\]

\( i) \ T_{PH}(p) \) is \( t \% \) true in \( PH \) where \( t \in T \).

\( ii) \ I_{PH}(p) \) is \( i \% \) true in \( PH \) where \( i \in I \).

\( iii) \ F_{PH}(p) \) is \( f \% \) true in \( PH \) where \( f \in F \).

The result \( t+i+f=1 \) refers that it is possible like in the situation of classical and fuzzy logics. Also the result \( t+i+f<1 \) refers that it is possible like in the situation of intuitionistic logic and the result \( t+i+f>1 \) refers that it is possible like in the situation of paraconsistent logic [3].

**Example:** The probability of a patient to pass his surgery is “60% true” according to his doctor in the hospital, “25 or 30-35% false” according to his weak immunity, and “15 or 20% indeterminate” due to equipment in the hospital.

**The Complement Of a Neutrosophic Set:** If we have \( U \) as an initial universe and if we take \( PH \) as a neutrosophic subset of \( U \), then the complement of \( PH \) is indicated as \( \langle PH \rangle \) and is defined as the following way:
The Containment Of Two Neutrosophic Sets: If we have $U$ as an initial universe and if we take $PH, SH$ as two neutrosophic subsets of $U$, we say $PH$ is contained in $SH$ and indicated by $PH \subseteq SH$, precisely when following holds:

$$T_{SH}(k) \leq T_{PH}(k)$$
$$I_{SH}(k) \leq I_{PH}(k)$$
$$F_{SH}(k) \geq F_{PH}(k)$$

$$\forall k \in U, T(k)_{PH}, I(k)_{PH}, F(k)_{PH} \in [0,1]$$

(4)

The Union Of Two Neutrosophic Sets: If we have $U$ as an initial universe and $PH, SH$ are two subsets of $U$, the union of these neutrosophic sets $PH$ and $SH$ will be a neutrosophic set $PS$, and we write $PS = PH \cup SH$, if and only if the conditions holds:

$$T_{PS}(k) = \max(T_{PH}(k), T_{SH}(k))$$
$$I_{PS}(k) = \max(I_{PH}(k), I_{SH}(k))$$
$$F_{PS}(k) = \min(F_{PH}(k), F_{SH}(k))$$

$$\forall k \in U, T(k)_{PH}, I(k)_{PH}, F(k)_{PH} \in [0,1]$$

(4)
The Intersection Of Two Neutrosophic Sets: If we have $U$ as an initial universe and if we take $PH, SH$ as two subsets of $U$, the intersection of this neutrosophic sets $PH$ and $SH$ will be neutrosophic set $PS$ and we write $PS = PH \cap SH$, if and only if the conditions holds:

\[
T_{PS}(k) = \min(T_{PH}(k), T_{SH}(k))
\]
\[
I_{SP}(k) = \min(I_{PH}(k), I_{SH}(k))
\]
\[
F_{PS}(k) = \max(F_{PH}(k), F_{SH}(k))
\]

$\forall k \in U, T(k)_{PH}, I(k)_{PH}, F(k)_{PH} \in [0,1]$

$\forall k \in U, T(k)_{SH}, I(k)_{SH}, F(k)_{SH} \in [0,1]$

(4)

The Difference of Two Neutrosophic Sets: If we have $U$ as an initial universe and if we take $PH, SH$ as two subsets of $U$, the difference of these neutrosophic sets $PH$ and $SH$ will be neutrosophic set $PS$ and we write $PS = PH - SH$, if and only if the conditions holds:

\[
T_{PS}(k) = \min(T_{PH}(k), F_{SH}(k))
\]
\[
I_{PS}(k) = \min(I_{PH}(k), 1 - I_{SH}(k))
\]
\[
F_{PS}(k) = \min(F_{PH}(k), T_{SH}(k))
\]

$\forall k \in U, T(k)_{PH}, I(k)_{PH}, F(k)_{PH} \in [0,1]$

$\forall k \in U, T(k)_{SH}, I(k)_{SH}, F(k)_{SH} \in [0,1]$

(4)

Example: If we have $U$ as an initial universe and if we take $PH, SH$ as two neutrosophic sets of $U$ like the following:

\[
PH = \{ \{k_1(0.3,0.5,0.6)\}, \{k_2(0.3,0.2,0.4)\}, \{k_3(0.8,0.7,0.2)\} \}
\]

\[
SH = \{ \{k_1(0.6,0.1,0.2)\}, \{k_2(0.3,0.2,0.6)\}, \{k_3(0.4,0.1,0.5)\} \}
\]

where $k_1, k_2, k_3 \in [0,1]$

then:

1) $\left( PH \right)^c = \{ \{k_1(0.6,0.5,0.3)\}, \{k_2(0.4,0.8,0.3)\}, \{k_3(0.2,0.3,0.8)\} \}$
we notice that $PHG$ is a commutative neutrosophic group if $g_1 \cdot g_2 = g_2 \cdot g_1$ for all $g_1, g_2 \in PHG$.

Theorem: If we have $PHG$ as a neutrosophic group, then $PHG$ always is a group but it must have a group[5].

Example 1: Under multiplication modulo 3 we have $\mathbb{Z}_3$ as a group but $PH\mathbb{Z}_3$ is not a group.

Theorem: If we have $PHG$ as a neutrosophic group, then $PHG$ always is a group but it must have a group[5].

Example 2: The groups $(PHZ, +), (PHQ, +)$ and $(PHC, +)$ are neutrosophic groups with $(+)$ [6].

Example 4: If we have the group $\{e, g_1, g_2, g_3\}$ with matrix multiplication modulo 3, then under multiplication $PHG$ is a commutative group [6].

Example 3: If we have the group $\{e, g_1, g_2, g_3\}$ with matrix multiplication modulo 3, then under multiplication $PHG$ is a commutative group [6].

Example 5: If we have the group $\{e, g_1, g_2, g_3\}$ with matrix multiplication modulo 3, then under multiplication $PHG$ is a commutative group [6].

Neutrosophic Groups

1. $SHG$ is not empty.
2. $SHG$ itself is a neutrosophic group.
3. $SHG$ must have a subset which is a group[5].

Example: If we have the neutrosophic group $PHG = \{e, g_1, g_2, g_3, I, g_4, I, g_5, I, g_6, I, g_7, I, g_8, I, g_9, I, g_{10}, I, g_{11}, I\}$ then the subgroups $SHG = \{e, g_1, I, g_2, I\}$, $NHG = \{e, g_2, I, g_4, I\}$ and $KHG = \{e, g_3, I, g_5, I\}$ are neutrosophic subgroups of $PHG$ [6].

The Order Of Neutrosophic Group

If we have $PHG$ as a neutrosophic group. The order of $PHG$ is the cardinal number $|PHG|$. $PHG$ is known as finite (resp. infinite) if $|PHG|$ is finite (resp. infinite) [5].

The Neutrosophic Element And Free Element

If we have $PHG$ as a neutrosophic group, and we have an element $k \in PHG$ which is called a neutrosophic element if is $t \in \mathbb{Z}^*$ it makes $k = I$ exist, and if $k \in \mathbb{Z}^*$ does not exist so $k$ is called a neutrosophic free element [5].

Example: If we have $PHG = \{1, 2, 5, I, 3I, 9I, 11I\}$ under multiplication modulo 10, then:

1. $|PHG| = 7$
2. $(1), (3I), (9I), (11I)$, neutrosophic elements.
3. $1, 2, 5$ a free neutrosophic elements.

Neutrosophic Rings

If we have $(R, +, \cdot)$ as any ring and $PH(R)$ neutrosophic set formed by $R$ and $I$, we define the triple $PH(R) = (R(I), +, \cdot)$ as a neutrosophic ring [5].
• If \( r S = S r \) for any \( r, s \in PH(R) \) we notice that it is a commutative neutrosophic ring.
• It is obvious \( R \subseteq PH(R) \).

**Example**: \( PHZ = \{t_1 + t_2 I | t_1, t_2 \in Z\} \) is a ring known as the commutative neutrosophic ring of integers [5].

**Example**: If we have \( (Q, +, \cdot) \) as a ring of rational numbers, then \( PHQ = \{q_1 + q_2 I | q_1, q_2 \in Q\} \) is the ring known as the commutative neutrosophic ring of rational numbers [5].

**Example**: If we have \( (R, +, \cdot) \) as a ring of real numbers, then \( PHR = \{r_1 + r_2 I | r_1, r_2 \in R\} \) is the neutrosophic ring known as the commutative neutrosophic ring of real numbers [5].

Example: If we have \( PH(R) \) as a neutrosophic ring, and if there is at least \( t \) like \( tr = 0, t \in Z^{+} \) for all \( r \in PH(R) \), then \( PH(R) \) is said to have characteristic \( t \). It is possible only if \( t = 0 \), then \( PH(R) \) is said to have characteristic zero.

**The Neutrosophic Ring of Characteristic**

If we have \( PH(R) \) as a neutrosophic ring, and if there is at least \( t \) like \( tr = 0, t \in Z^{+} \) for all \( r \in PH(R) \), then \( PH(R) \) is said to have characteristic \( t \). It is possible only if \( t = 0 \), then \( PH(R) \) is said to have characteristic zero.

**Neutrosophic Subrings**

If we have \( PH(R) \) as a neutrosophic ring and if we take \( SH(R) \) as a subset in \( PH(R) \), we define \( SH(R) \) as a neutrosophic subring precisely when:

1) \( SH(R) \neq \emptyset \)
2) \( SH(R) \) itself is a neutrosophic ring.
3) \( SH(R) \) must has a proper subset which is a ring [5].

**Example**: The set \( SHE \) of even neutrosophic integers is a commutative subring of integers neutrosophic ring \( PHZ \).

**Neutrosophic Ideals**

If we have \( PH(R) \) as a neutrosophic ring. A neutrosophic subring \( SH(J) \) of \( PH(R) \) is said to be an ideal of \( PH(R) \) precisely when: \( \forall r \in PH(R), \forall j \in SH(J) \Rightarrow jr, jr \in SH(J) \).

**Example**: \( SH(2Z) \) is a neutrosophic ideal of \( PH(Z) \).

**Neutrosophic Fields**

If we have \( (F, +, \cdot) \) as a field and if we take \( PH(F) \) as a set formed by \( F \) with \( I \), we will define the form \( PH(F) = \{F(I), +, \cdot\} \) as a neutrosophic field [5].

• For \( \alpha \in F \) then \( \alpha I = \alpha I \), \( 0 \in PHF \) is formed by \( 0 + 0 I \), \( 0 I = 0 \) and \( 1 \in PHF \) is formed by \( 1 + 0 I \) in \( PHF \).

**Example 1**: \( PH(Q) \) is the neutrosophic field of rational numbers which formed by rational numbers and \( I \) [5].

**Example 2**: \( PH(R) \) is the neutrosophic field of real numbers which formed by real numbers and \( I \) [5].

**Example 3**: \( PH(C) \) is the neutrosophic field of complex numbers which formed by complex numbers and \( I \) [5].

**Neutrosophic Subfields**: If we have \( PH(F) \) as a neutrosophic field and if we take \( SP(F) \) as a subset of of \( PH(F) \), we will define \( SP(F) \) as a neutrosophic subfield precisely when:

1) \( SP(F) \neq \emptyset \).
2) \( SP(F) \) itself is a neutrosophic field.
3) \( SP(F) \) must has a subset which is a field [5].

**Example**: \( PH(R) \) is neutrosophic subfield of \( PH(C) \).

**Neutrosophic Vector Spaces**

If we have \( (V, +, \cdot) \) as a vector space over \( F \) and if we take the set \( PH(V) \) which formed by \( V \) and \( I \), we will define the form
as a weak neutrosophic vector space (WNV) over \( F \). However, if \( F \) is a \( PH(F) \) then \( (PH(V), +, \cdot) \) is defined as a strong neutrosophic vector space (SNV) over \( PH(F) \) [3].

**Example:** We can look at \( PH(R) \) through two views it's a (WNV) over \( Q \). Moreover, it's a (SNV) over \( PH(Q) \) [3].

**Neutrosophic Subspace:** If we have \( PH(V) \) as a (WNV) over \( F \) and if we take \( (SHV) \) as a subset of \( PH(V) \), we will define \( (SHV) \) as a (WNV) subspace of \( PH(V) \) precisely when:

1) \( (SH(V)) \neq \emptyset \)
2) \( (SHV) \) itself a weak neutrosophic vector space over \( F \).
3) \( (SHV) \) must has a subset which is a vector space [3].

- Similarly for (SNV) subspace.

**Example 1:** If we have \( PH(V) \) as (WNV) or (SNV), then \( PH(V) \) is a subspace of itself and known as a trivial (WNV) or (SNV) subspace [3].

**Example 2:** If we have \( PH(V) \) as a (SNV) over \( PH(R) \). And if we take the subset \( (SHV) \) as a (SNV) subspace of \( PH(V) \) [3].

**Neutrosophic Left Module**

If we have \( (sM, +, \cdot) \) as a left \( R \) – module over a ring \( R \) and we have \( PH(M) \) as a neutrosophic non-empty set formed by \( sM \) and \( I \), then the triple \( PH(sM) = (sM(I), +, \cdot) \) is defined as a weak neutrosophic left \( R \) – module (WNML) over \( R \).

- If \( R \) is \( PH(R) \), then \( PH(M_{p+q}) \) is defined as a strong neutrosophic left \( R \) – module (SNML) over \( PH(R) \) [7-12].

**Neutrosophic Right \( R \) – Module**

Similarly the form \( PH(M) = (M(I), +, \cdot) \) is known as weak neutrosophic right \( R \) – module (WNML) over a ring \( R \).

- If \( R \) is \( PH(R) \), then \( PH(M_{p+q}) \) is defined as a strong neutrosophic right \( R \) – module (SNMR) over \( PH(R) \).

**Notes:**

1) If we have \( R(PH(R)) \) as a commutative ring (neutrosophic ring), then every \( PH(M) \) is a \( PH(M_{p+q}) \) (PH(M_{p+q})).
2) In general neutrosophic \( R \) – module (strong , weak , lift and, right ) denoted by \( PH(M) \) or (NM).
3) If \( \{m,n \in PH(M) : m=a+bl, n=k+tI \} \) where \( a,b,k \) and \( t \) are elements in \( M \) and \( \{r \in R(I) : r=p+ql \} \) where \( P \) and \( q \) are scalars in \( R \) we define:

\[
\begin{align*}
\text{i) } m + n &= (a+bl) + (k+tl) \\
&= (a+k) + (b+t)I \\
\text{ii) } rm &= (p+ql)(a+bl) \\
&= pa + (pb+qa+qb)I
\end{align*}
\]

**Example 1:** If we have \( PH(R) \) as a (NR) and if we take \( PH(J) \) as a (NI) of \( PH(R) \), then:

1) \( PH(R) \) is a \( PH(M) \).
2) \( PH(J) \) is a neutrosophic \( R \) – module under the addition and multiplication of \( PH(R) \).
3) \( PH(R)/PH(J) \) is a \( PH(M) \).

**Example 2:** We can look at \( PH(R^2) \) through two views: the first is a (WNM) over a ring \( R \), and the second is a (SNM) over a neutrosophic ring \( PH(R) \).

**Example 3:** We can look at \( PH(M_{p+q}) \) through two views: the first is a (WNV) over a ring \( Q \), and
The second is a (SNM) over $PH(Q)$.

**Theorem 4:** If we have a (SNM) then it is a (WNM).

**Proof:** If we have $PH(M)$ as a (SNM) because it is $R \subseteq PH(R)$ for every ring $R$, then $PH(M)$ is a (WNM) over $R$.

**Theorem:** If we have $PH(M)$ as a commutative group then every weak (strong) neutrosophic $R-$ module is a $R-$ module.

**Proof:** If we have $m=a+bl, n=k+tl \in PH(M)$ where $a,b,k,t \in M$ and $a=p+ql, b=r+sl \in PH(R)$, $p,q,r,s \in R$ then:

1) $a(m+n) = (p+ql)(a+bl+k+tl)$

   $= pa + pk + (pb + pt + qa + qb + qk + qt)I$

   $= \alpha m + \alpha n$

2) $(\alpha + \beta)m = (p+ql+r+sl)(a+bl)$

   $= pa + ra + (qp + rb + sa + sb)I$

   $= \alpha m + \beta m$

3) $(\alpha \beta)m = ((p+ql)(r+sl))(a+bl)$

   $= pra + (prb + psa + psb + qra + qrb + qsa + qsb)I$

   $= \alpha (\beta m)$

4) For $1 + 0I \in PH(R)$ we have:

   $1m = (1 + 0I)(a+bl)$

   $= 1a + 1bl + 0aI + 0bf^2$

   $= a + bl + 0 + 0 = a + bl = m$ So that, $PH(M)$ is a $R-$ module.

**Lemma:** If we have $PH(M)$ as a neutrosophic $R-$ module over ring $PH(R)$ and if we take $m=a+bl, h=k+tl, s=e+fl \in PH(M)$, $\alpha = p+ql \in PH(R)$ then:

1) $m+s = s+h \Rightarrow m = s$

2) $\alpha 0 = 0$

3) $0m = 0$

4) $(-\alpha)m = \alpha (-m) = -(\alpha m)$

**Neutrosophic Submodule**

If we have $PH(M)$ as a neutrosophic $R-$ module over a ring $PH(R)$ and if we take $SH(N)$ as a subset of, we will define as a strong neutrosophic submodule precisely when:

1) $SH(N) \neq \emptyset$

2) $SH(N)$ itself is a strong neutrosophic $R-$ module.

3) $SH(N)$ must has a proper subset which is a $R-$ module.
• Similarly for weak neutrosophic submodule.

**Theorem:** If we have $PH(M)$ as a neutrosophic $R$-module over a ring $PH(R)$ and if we take $SH(N)$ as a subset of $PH(R)$, we will define as a strong neutrosophic submodule precisely when:

1) $SH(N) \neq \emptyset$

2) $n, n' \in SH(N) \Rightarrow n + n' \in SH(N)$.

3) $n \in SH(N), \alpha \in PH(R) \Rightarrow \alpha n \in SH(N)$.

4) $SH(N)$ must have a proper subset which is a $R$-module.

**Corollary:** If we have $PH(M)$ as a (NM) over a ring $PH(R)$ and if we take $SH(N)$ as a subset of $PH(M)$, then $SH(N)$ is neutrosophic submodule of $PH(M)$ precisely when:

1) $SH(N) \neq \emptyset$

2) $\forall h, t \in SH(N), \forall r, q \in PH(R) \Rightarrow rh + qt \in SH(N)$.

3) $SH(N)$ must have a proper subset which is a $R$-module.

**Example 1:** If we have $PH(M)$ as a (NM) over a ring $PH(R)$, then $PH(M)$ is a neutrosophic submodule known as a trivial neutrosophic submodule.

**Example 2:** If we have $PH(M)$ as a (NM) over a ring $PH(R)$ and if we take $SH(N)$ as a subset of $PH(M)$, then $SH(N)$ is neutrosophic submodule of $PH(M)$ precisely when:

1) $SH(N) \neq \emptyset$

2) $\forall h, t \in SH(N), \forall r, q \in PH(R) \Rightarrow rh + qt \in SH(N)$.

3) $SH(N)$ must have a proper subset which is a $R$-module.

**Theorem:** If we have $PH(M)$ as a (NM) over a ring $PH(R)$ and if we take $\{SH(N)\}_{i=1}^n$ as a set of all neutrosophic submodule of $PH(M)$, then $\bigcap_{i=1}^n SH(N)$ is neutrosophic submodule of $PH(M)$.

**Proof:** It’s clearly $\bigcap_{i=1}^n SH(N) \neq \emptyset$, then:

1) If we have $k, t \in \bigcap_{i=1}^n SH(N)$, then $k - t \in \bigcap_{i=1}^n SH(N)$.

2) If we have $h \in SH(N)$ and $\alpha \in PH(R)$, then $\alpha h \in \bigcap_{i=1}^n SH(N)$.

Since for $\forall i \in A$ implies $\bigcap_{i=1}^n SH(N)$ is neutrosophic submodule of $PH(M)$.

**Remark:** If we have $PH(M)$ as a (NM) over a ring $PH(R)$, and $SH(A), SH(B)$ as two different neutrosophic submodules of $PH(M)$. In general, $SH(A) \cup SH(B)$ is not a neutrosophic submodule of $PH(M)$. However, if $SH(A) \subseteq SH(B)$ or $SH(B) \subseteq SH(A)$ true, then $SH(A) \cup SH(B)$ is a neutrosophic submodule of $PH(M)$.

**The Sum And Direct Sum Of Two Neutrosophic Submodule**

If we have $SH(A)$ and $SH(B)$ as two neutrosophic submodules of $PH(M)$ over a neutrosophic ring $PH(R)$ then:

1) We define the sum of $SH(A)$ and $SH(B)$ by the set:

$$\{n_1 + n_2 : n_1 \in SH(A), n_2 \in SH(B)\}$$

and refer by $SH(A) + SH(B)$.

2) $PH(M)$ is said to be the direct sum of $SH(A)$ and $SH(B)$ precisely when:

$$\forall m \in PH(M) \Rightarrow m = n_1 + n_2$$

where $n_1 \in SH(A)$ and $n_2 \in SH(B)$.

We denoted by $PH(M) = SH(A) \oplus SH(B)$.

**Example:** If we have $PH(M) = PH(R)$ as a (NM) over a ring $PH(R)$ and if we take $SH(A), SH(B)$ as two neutrosophic submodules of $PH(M)$ like:

$$SH(A) = \{(n_1, 0, 0) : n_1 \in PH(R)\}$$
\[ SH(B) = \{(0, n_2, n_3) : n_1, n_2 \in PH(R)\} \]

Then \( PH(M) = SH(A) \oplus SH(B). \)

**Lemma:** If we have \( SH(A) \) as a neutrosophic submodule of a neutrosophic \( R \)-module \( PH(M) \) over neutrosophic ring \( PH(R) \), then:

1) \( SH(A) + SH(A) = SH(A) \)
2) \( n + SH(A) = PH(A), \ \forall n \in SH(A). \)

**Theorem:** If we have \( SH(A) \) and \( SH(B) \) as two neutrosophic submodules of \( PH(M) \) over a neutrosophic ring \( PH(R) \), then:

1) \( SH(A) + SH(B) \) is a neutrosophic submodule of \( PH(M) \).
2) \( SH(A) \) and \( SH(B) \) are contained in \( SH(A) + SH(B) \).

**Proof:**
1) Clearly, \( A + B \) is a submodules contained in \( SH(A) + SH(B) \).

Let \( n, k \in SH(A) + SH(B) \) and let \( \alpha, \beta \in PH(R) \). Then:

\[
\begin{align*}
\alpha n + \beta k &= [(an_1 + cn_3) + (an_2 + bn_1 + cn_4 + dn_3 + dn_4)I] + [ak_1 + ck_3] + [ak_2 + bk_1 + ck_4 + dk_3 + dk_4)I] \\
&\Rightarrow \alpha n + \beta k \in SH(A) + SH(B)
\end{align*}
\]

Accordingly, \( SH(A) + SH(B) \) is a neutrosophic submodule of \( PH(M) \).

2) Clear.

**Theorem:** If we have \( SH(A) \) and \( SH(B) \) as two neutrosophic submodules of \( PH(M) \) over a neutrosophic ring \( PH(R) \), then \( PH(M) = SH(A) \oplus SH(B) \) if and only if:

1) \( PH(M) = SH(A) + SH(B) \)
2) \( SH(A) \cap SH(B) = \{0\} \).

**Theorem:** If we have \( SH(A) \) and \( SH(B) \) as two neutrosophic submodules of \( PH(M) \) over a neutrosophic ring \( PH(R) \), then \( SH(A) \times SH(B) = \{(n_1, n_2) : n_1 \in SH(A), n_2 \in SH(B)\} \) is a neutrosophic \( R \)-module over a ring \( PH(R) \) where addition and multiplication are defined by the form:

1) \( (m_1, m_2) + (n_1, n_2) = (m_1 + n_1, m_2 + n_2) \)
2) \( \alpha(m_1, m_2) = (\alpha m_1, \alpha m_2) \)

**The Neutrosophic Quotient \( R \)-module**

If we have \( SH(N) \) as a neutrosophic submodule of a neutrosophic \( R \)-module \( PH(M) \) over a ring \( PH(R) \), then the quotient \( PH(M)/SH(N) \) is defined as \( [n + SH(N) : n \in PH(M)] \) which can be made a neutrosophic \( R \)-module over a ring \( PH(R) \). We can define the addition and multiplication as in the following way:
The neutrosophic $R -$ module $PH(M) / SH(N)$ over a ring $PH(R)$ is defined as a neutrosophic quotient $R -$ module.

Example: If we have $PH(M)$ as a neutrosophic $R -$ module over a ring $PH(R)$, then $PH(M) / PH(M)$ is a neutrosophic zero $R -$ module.

Neutrosophic Linear Combination , Neutrosophic Linearly Independent Set And Neutrosophic Linearly Dependent Set

If we have $PH(M)$ as a neutrosophic $R -$ module over a ring $PH(R)$ , then:

1) An element $m \in PH(M)$ is known as a linear combination of the $\{ m_1, m_2, ..., m_n \}$ if $\{ \alpha_1, \alpha_2, ..., \alpha_n \} \in PH(R)$.

2) If $\alpha_1 m_1 + \alpha_2 m_2 + ... + \alpha_n m_n = 0 \Rightarrow \alpha_1 = \alpha_2 = ... = \alpha_n = 0$ ( all $\alpha$ are equal to zero), then, $\{ m_1, m_2, ..., m_n \}$ is defined as a linearly independent set.

3) $\{ m_1, m_2, ..., m_n \}$ are said to be linearly dependent if $\alpha_1 m_1 + \alpha_2 m_2 + ... + \alpha_n m_n = 0$ implies that not all $\alpha$ are equal to zero, then $\{ m_1, m_2, ..., m_n \}$ is defined as a linearly dependent set.

Example: If we have $PH(M)$ as a neutrosophic $R -$ module over a ring $PH(R)$, an element $m = 8 + 4l \in PH(M)$ is a linear combination of the elements $m_1 = 1 + 21, m_2 = 2 + 3l \in PH(M)$ since $8 + 4l = -16(1 + 21) + 12(2 + 3l)$.

Theorem: If we have $PH(M)$ as a neutrosophic $R -$ module over a ring $PH(R)$, then we can infinitely explained as a linear combination of the $\{ m_1, m_2, ..., m_n \}$.

Proof: if we have $m = m_1 \alpha_1 + m_2 \alpha_2 + ... + m_n \alpha_n$

where $m = a + bl, m_1 = a_1 + b_1 l, m_2 = a_2 + b_2 l, ..., m_n = a_n + b_n l \in PH(M)$

and $\alpha_1 = x_1 + y_1 l, \alpha_2 = x_2 + y_2 l, ..., \alpha_n = x_n + y_n l \in PH(R)$

Then: $a + bl = (x_1 + y_1 l)(a_1 + b_1 l) + (x_2 + y_2 l)(a_2 + b_2 l) + ... + (x_n + y_n l)(a_n + b_n l)$ from which we have

$$a_1 x_1 + a_2 x_2 + ... + a_n x_n = a$$

$$b_1 x_1 + a_1 y_1 + b_1 y_1 + b_2 x_2 + a_2 y_2 + ... + b_n x_n + a_n y_n + b_n y_n = b$$
That means there are many solution of \( x_1, y_2, t = 1, 2, ..., n \). It implies that the \( \{ m_1, m_2, ..., m_n \} \) can be infinitely combined to produce.

**Notice:** If we have \( PH(M) \) as a neutrosophic \( R \) – module over a ring \( PH(R) \), and we have \( am = 0 \), this does not mean \( a = 0 \in PH(R) \) or \( m = 0 \in PH(M) \)

all the times it is possible to have \( m \neq 0 \) and \( a \neq 0 \). For example, if \( m = x - xI, m \in PH(M), x \neq 0 \) and \( a = yI \) where \( a \in PH(R) \), \( x, y \in R \) we have \( am = yI(x + xI) = ydI - ydI = 0 \).

**Theorem:** If we have \( PH(M) \) as a neutrosophic \( R \) – module over a ring

\( PH(R) \) and \( m_1 = x_1 - x_1I, m_2 = x_2 - x_2I, ..., m_n = x_n - x_nI \in PH(M) \) where \( x_1 \neq 0 \in R \), then \( \{ m_1, m_2, ..., m_n \} \) is a linearly dependent set.

**Proof:** If we have \( \alpha_1 = p_1 + q_1I, \alpha_2 = p_2 + q_2I, ..., \alpha_n = p_n + q_nI \in PH(R) \), then \( \alpha_1m_1 + \alpha_2m_2 + ... + \alpha_nm_n = 0 \)

\( \Rightarrow (p_1 + q_1I)(x_1 + x_1I) + (p_2 + q_2I)(x_2 + x_2I) + + (p_n + q_nI)(x_n + x_nI) = 0 \)

\( \Rightarrow x_1p_1 + x_2p_2 + ... + x_np_n = 0 \)

That means there are many nontrivial solution of \( p_i : i = 1, 2, ..., n \). As a result, \( \{ m_1, m_2, ..., m_n \} \) is a linearly dependent set.

**Example:** If we have \( PH(M) = PH(R^n) \) as a neutrosophic \( R \) – module over a ring \( PH(R) \). As an example of a linearly independent set in \( PH(M) \) we can take the set \( \{ m_1 = (1,0,0,...,0), m_2 = (0,1,0,...,0), ..., m_n = (0,0,...,1) \} \).

**Theorem:** If we have \( PH(M) \) as a neutrosophic \( R \) – module over a ring \( PH(R) \) and if we take \( SH(N) \) as a nonempty subset of \( PH(M) \), we refer to the all linear combinations of \( SH(N) \) by the form \( \xi(SH(N)) \), then:

1) \( \xi(SH(N)) \) is a (SN) submodule of \( PH(M) \) having \( SH(N) \).

2) If we have \( SH(A) \) as a (SN) submodule of \( PH(M) \) having \( SH(N) \), then \( \xi(SH(N)) \subseteq SH(A) \).

**Proof:** 1) Clearly, because \( SH(N) \) is a nonempty set, as a result \( \xi(SH(N)) \) is nonempty. \( \forall m = x + yI \in SH(N), \alpha = 1 + 0I \), if we write

\( \alpha m = (1 + 0I)(x + yI) \)

\( = x + yI \in \xi(SH(N)) \)

\( \Rightarrow SH(N) \subseteq \xi(SH(N)) \)

Finally, \( \forall m, n \in \xi(SH(N)), m_i, n_i \in SH(N), \alpha, \beta_i \in PH(R) \) then:

\( m = \alpha_1m_1 + \alpha_2m_2 + ... + \alpha_nm_n \)

\( n = \beta_1n_1 + \beta_2n_2 + ... + \beta_nm_n \)

\( \Rightarrow \alpha m + \beta n \in \xi(SH(N)) \) For \( \alpha, \beta \in PH(R) \). Since \( SH(N) \) is a subset of \( \xi(SH(N)) \) which is a submodule of \( PH(M) \) having \( SH(N) \), \( \xi(SH(N)) \) is a neutrosophic submodule of \( PH(M) \) having \( SH(N) \).

2) As it is in the classical case and it is cancelled.

**The Span Of Neutrosophic module**

If we have \( PH(M) \) as a neutrosophic \( R \) – module over a ring \( PH(R) \), the neutrosophic submodule \( \xi(SH(N)) \) of theorem 3.5.5 is defined as the span of \( SH(N) \) and it is referred to as \( \text{span}SH(N) \).
• If \( PH(M) = \text{span}SH(N) \), then we say \( SH(N) \) a span \( PH(M) \).

The Basis Of Neutrosophic module

If we have \( PH(M) \) as a neutrosophic \( R \) – module over a ring \( PH(R) \) and \( SH(N) = \{n_1, n_2, \ldots, n_n\} \) as a linearly independent subset of \( PH(M) \), the subset \( SH(N) \) is defined as a basis for \( PH(M) \) precisely when \( SH(N) \) is a span \( PH(M) \).

Example: If we have \( PH(M) = PH(R') \) as a neutrosophic \( R \) – module over a Ring \( PH(R) \), and if we take the sub set:

\[
SP(N) = \{n_1 = (I, 0, 0), n_2 = (0, I, 0), \ldots, n_n = (0, 0, I)\}
\]

we can find it as a basis for \( PH(M) \).

Free Neutrosophic module

If we have \( PH(M) \) as a neutrosophic \( R \) – module over a ring \( PH(R) \), \( PH(M) \) is said to be free neutrosophic module when it has a basis.

for examples:

• \( PH(R) \) itself is a free neutrosophic \( R \) – module, having a basis element \( 1_{PH(R)} \).
• The zero neutrosophic \( R \) – module \( 0 \) is a free neutrosophic \( R \) – module with empty basis.
• \( PH(M) = PH(\mathbb{R}) \) is a free neutrosophic \( R \) – module with basis \( SH(N) \) in Example 3.5.5.

Theorem: If we have \( PH(M) \) as a neutrosophic \( R \) – module over a ring \( PH(R) \), the basis of \( PH(M) \) over \( PH(R) \) are like the basis of \( M \) over a \( R \).

Proof: Let \( N = \{n_1, n_2, \ldots, n_n\} \) be any basis for \( M \) over \( R \). We will proof \( N \) is a basis of \( PH(M) \):

1) We will show \( N \) is a linearly independent set in \( PH(M) \): let \( \alpha_1 = k_1 + m_1I, \alpha_2 = k_2 + m_2I, \ldots, \alpha_n = k_n + m_nI \in PH(R) \), if

\[
\alpha_1 + k_1n_1 + k_2n_2 + \ldots + k_nn_n = 0 \Rightarrow k_1 = k_2 = \ldots = k_n = 0, m_1 = m_2 = \ldots = m_n = 0
\]

We have \( k_j = 0 \) and \( m_j = 0 \) where \( i, j = 0, 1, 2, \ldots, n \Rightarrow \alpha_i = 0, i = 1, 2, \ldots, n \). This explains that \( N \) is a linearly independent set in \( PH(M) \).

2) We will show \( N \) that span \( PH(M) \):

let \( n = a + bI = \alpha_1n_1 + \alpha_2n_2 + \ldots + \alpha_n n_n \). Then we have \( a = k_1n_1 + k_2n_2 + \ldots + k_nn_n, b = m_1n_1 + m_2n_2 + \ldots + m_n n_n \)

\( a, b \in M \Rightarrow n = a + bI \) can be formed uniquely as a linear combination of \( \{n_1, n_2, \ldots, n_n\} \).

Theorem: If we have \( PH(M) \) as a \( (NM) \) over ring \( PH(R) \), then the basis of strong \( (NM) \) is contained in the basis of the weak \( (NM) \).

Neutrosophic Module Homomorphism

If we have \( PH(M) \) and \( PH(N) \) as two neutrosophic \( R \) – modules over a ring \( PH(R) \), a mapping \( \varphi: PH(M) \rightarrow PH(N) \) (PH(M) into \( PH(N) \)) is said to be a neutrosophic \( R \) – module homomorphism (NMH), precisely when:

1) \( \varphi(rm + r'm') = r\varphi(m) + r'\varphi(m') \) for all \( m, m' \in M \) and \( r, r' \in R \).

2) \( \varphi(I) = I \).

Remarks:

• \( \varphi \) is said to be a neutrosophic \( R \) – module monomorphism precisely when it is one-one.
• \( \varphi \) is said to be a neutrosophic \( R \) – module epimorphism precisely when it is onto.
• \( \varphi \) is said to be neutrosophic \( R \) – module isomorphism precisely when it is one-one and onto.

If \( \varphi \) is a neutrosophic \( R \) – module isomorphism, then the invers \( \varphi^{-1}: PH(N) \rightarrow PH(M) \) is also neutrosophic \( R \) – module isomorphism and we write \( PH(M) \cong PH(N) \).

Example: The mapping \( PH(M) \cong PH(N) \) defined by \( z(m) = 0_{PH(N)} \) for all \( m \in PH(M) \) is a neutrosophic \( R \) – module homomorphism since \( I \in M(I) \) but \( z(I) \neq 0 \), called the zero neutrosophic \( R \) – module homomorphism.
The Kernel And The Image Of Neutrosophic \( R \)- Module Homomorphism

If we have \( PH(M) \) and \( PH(N) \) as two neutrosophic \( R \)-modules over a ring \( PH(R) \), and if we take \( \varphi : PH(M) \rightarrow PH(N) \) as a neutrosophic \( R \)-module homomorphism, then:

- The kernel of \( \varphi \) referred to as \( \ker \varphi \) is defined by the set \( \ker \varphi = \{ m \in PH(M) : \varphi(m) = 0 \} \).
- The image of \( \varphi \) referred to as \( \text{Im}(\varphi) \) is defined by the set \( \text{Im}(\varphi) = \{ n \in PH(N) : \varphi(m) = n \text{ for } m \in PH(M) \} \).

**Example:** If we have \( PH(M) \) and \( PH(N) \) as two neutrosophic \( R \)-modules over a ring \( PH(R) \) and if we take \( \varphi : PH(M) \rightarrow PH(N) \) defined by the form \( \forall m \in PH(M) : \varphi(m) = m \), then:

1) \( \varphi \) is a neutrosophic \( R \)-module homomorphism.
2) \( \ker \varphi = \{0\} \).
3) \( \text{Im}(\varphi) = PH(M) \)

**Theorem:** If we have \( PH(M) \) and \( PH(N) \) as two neutrosophic \( R \)-modules over a ring \( PH(R) \) and if we take \( \varphi : PH(M) \rightarrow PH(N) \) as a neutrosophic \( R \)-module homomorphism, then:

1) \( \ker \varphi \) is not a neutrosophic submodule of \( PH(M) \) but a submodule of \( M \).
2) \( \text{Im} \varphi \) is a neutrosophic submodule of \( PH(M) \).

**Proof:**
1) In order for \( \ker \varphi \) to be a strong neutrosophic submodule of \( PH(M) \) it must contained \( I \) but \( \varphi(I) \neq 0 \) that means \( I \notin \ker \varphi \). As we have in the classical case \( \ker \varphi \) is a submodule of \( M \).
2) It is clear from the definition of \( \text{Im} \varphi \).
3) If we have \( PH(M) \) as a neutrosophic \( R \)-module over a ring \( PH(R) \), then the basis of \( PH(M) \) over \( PH(R) \) are like the basis of \( M \) over \( R \).

We can notes:

1) \( \sigma \) is a neutrosophic \( R \)-module homomorphism.
2) \( \ker \sigma = \ker \varphi \cap SH(A) \).
3) \( \text{Im} \sigma = \varphi \left( SH(A) \right) \).

**Remark:** If we have \( PH(M) \) and \( PH(N) \) as two neutrosophic \( R \)-modules over a ring \( PH(R) \) and if we take \( \sigma, \varphi : PH(M) \rightarrow PH(N) \) as two neutrosophic \( R \)-module homomorphisms, then: \( (\sigma + \varphi) \) and \( (\alpha \varphi) \) are not neutrosophic \( R \)-module homomorphisms.

**Proof:** Through the conditions of the neutrosophic \( R \)-module homomorphism, it must be \( (\varphi + \sigma)(I) = I \) but we have \( (\sigma + \varphi)(I) = \varphi(I) + \sigma(I) = I + I = 2I \neq I \) and must be \( (\alpha \varphi)(I) = \varphi(I) \) for any \( \alpha \in PH(R) \). Therefore, \( (\sigma + \varphi) \) and \( (\alpha \varphi) \) are not neutrosophic \( R \)-module homomorphisms.

**Note:** As a result of the remarks above, the set of all neutrosophic \( R \)-module homomorphisms from \( PH(M) \) into \( PH(N) \) is not neutrosophic \( R \)-module homomorphisms over \( PH(R) \) that means we have a different case from the classical \( R \)-module homomorphisms.
The Composition Of Tow Neutrosophic $R-$ Module Homomorphisms

If we have $PH(M), PH(N)$ and $PH(A)$ as three neutrosophic $R-$ modules over a ring $PH(R)$ and if $\varphi: PH(M) \to PH(N)$, $\sigma: PH(N) \to PH(A)$ as two (NMH), then the composition $\sigma \circ \varphi: PH(M) \to PH(A)$ is defined as:

$$\forall m \in PH(M) \Rightarrow (\sigma \circ \varphi)(m) = \sigma(\varphi(m)).$$

Note: $\sigma \circ \varphi: PH(M) \to PH(A)$ is also a neutrosophic $R-$ module homomorphism.

Proof: Through the conditions of the neutrosophic $R-$ module homomorphism, it must be $(\sigma \circ \varphi)(I) = I$. If we take $m = I \in PH(M)$, then:

$$(\sigma \circ \varphi)(I) = \sigma(\varphi(I)) = \sigma(I) = I$$

As a result $\sigma \circ \varphi$ is a neutrosophic $R-$ module homomorphism.

Corollary: If we have $\sigma, \varphi, \mu$ as three neutrosophic $R-$ module homomorphisms, from $PH(M)$ into $PH(M)$, then:

$$\sigma \circ (\varphi \circ \mu) = (\sigma \circ \varphi) \circ \mu$$

Theorem: If we have $PH(M), PH(N)$ and $PH(A)$ as three neutrosophic $R-$ modules over a ring $PH(R)$ and if $\varphi: PH(M) \to PH(N)$, $\sigma: PH(N) \to PH(A)$ as two neutrosophic module homomorphisms, then:

1) If $\sigma \circ \varphi$ is one-one, then $\varphi$ is one-one.
2) If $\sigma \circ \varphi$ is onto, then $\sigma$ is onto.
3) If $\sigma$ and $\varphi$ are one-one, then $\sigma \circ \varphi$ is one-one.

The Exact Sequence Of Neutrosophic $R-$ modules

If we have $PH(M), PH(N)$ and $PH(A)$ as three neutrosophic $R-$ modules over a ring $PH(R)$ and if $\varphi: PH(M) \to PH(N)$, $\sigma: PH(N) \to PH(A)$ as two (NMH), then we say that the Sequence $PH(M) \xrightarrow{\varphi} PH(N) \xrightarrow{\sigma} PH(A)$ is an exact sequence, precisely when $\text{Im} \varphi = \ker \sigma$.

Conclusion

In this thesis we inspired from the neutrosophic philosophy which F.Smarandanche introduced the theory of neutrosophy in 1995. Basically we defined neutrosophic R-modules and neutrosophic submodules which are completely different from the classical module and submodule in the structural properties. It was shown that every weak neutrosophic R-module is a R-module and every strong neutrosophic R-module is a R-module. Finally, neutrosophic quotient modules and neutrosophic R-module homomorphism are explained and some definitions and theorems are given.

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