Cubic Averaging Aggregation Operators with Multiple Attributes Group Decision Making Problem

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Abstract

The group decision making is a very useful technique for ranking the group of alternatives, the cubic averaging operator is a new tool in group decision making problems. In this article, we develop a series of new operators so called cubic weighted averaging (CWA) operator, cubic ordered weighted averaging (COWA) operator and cubic hybrid averaging (CHA) operator. We also discussed some particular cases and properties of these operators. Furthermore, we apply the proposed aggregation operators to deal with multiple attribute group decision making in which decision information takes the form of cubic numbers. Finally, we used some practical examples to illustrate the validity and feasibility of the proposed methods by comparing with other methods.

Keywords: Cubic numbers; Aggregation operators; Group decision making

Introduction

The idea of fuzzy set theory was developed by LA Zadeh [1]. In fuzzy set theory the degree of membership function was discussed. Fuzzy set theory has been studied in many directions such as medical diagnosis, computer science, artificial intelligence, operation research, management science, control engineering, robotics, expert systems and decision making problems. Later on the great idea of intuitionistic fuzzy set (IFS) theory was developed by Atanassov [2], and discussed the degree of membership and non-membership function. IFS is the generalization of fuzzy set theory. There are many advantages of IFS such as, using engineering, management science, computer science. Fuzzy set discuss the case in which the membership involves, intuitionistic fuzzy sets (IFS) is generalization of fuzzy set, the uncertainty problem does not explain by means of intuitionistic fuzzy set.

So therefore Jun et al. [3] define the concept of cubic set, Jun define a new theory which is known as cubic set theory. This theory is able to deal with uncertain problem. Cubic set theory also explains the, satisfied, unsatisfied, and uncertain information. While fuzzy set theory and intuitionistic fuzzy set fail to explain these terms, The cubic set is a generalization of fuzzy set and intuitionistic fuzzy set. Cubic set is collection of interval value fuzzy set (IVFS) and fuzzy set, while (IFS) is only fuzzy set. Cubic set more desirable the information then fuzzy set and intuitionistic fuzzy set. After that, Mahmood et al. [4] defined cubic hesitant fuzzy sets (CFHSs) by combining interval valued hesitant fuzzy sets (IVHFSs) (2013) and hesitant fuzzy sets (HFSs) (2009) and defined some basic operations, properties, of cubic hesitant fuzzy sets.

The concept of neutrosophic set (NS) developed by Smarandache [5] and Chang Su Kim and Smarandache [6] is a more general platform which extends the concepts of the classic set and fuzzy set, intuitionistic fuzzy set and interval valued intuitionistic fuzzy set. Neutrosophic set theory is applied to various part which extend the concept of cubic sets to the neutrosophic sets. They introduced the notions of truth-internal (indeterminacy-internal, falsity-internal) neutrosophic cubic sets and truth-external (indeterminacy-external, falsity-external) neutrosophic cubic sets, and investigate related properties. Such that the P-union and the P-intersection of truth-internal (indeterminacy-internal, falsity-internal) neutrosophic cubic sets are also truth-internal (indeterminacy-internal, falsity-internal) neutrosophic cubic sets. Jun et al. [7] also worked on neutrosophic cubic sets and developed it various properties.

Information aggregation is an important research topic in many applications such as fuzzy logic systems and multi-attribute decision making as discussed by Chen and Hwang [8]. Research on aggregation operators has received increasing attention as shown in the literature. Research papers of Yager and Kacprzyk, Yager [9-11], Calvo et al. [12], Xu and Da [13], Chen and Chen [14], and others are included in this area. Research on aggregation methods with intuitionistic fuzzy or intervalvalued intuitionistic
fuzzy information is also active. Mitchell [15] defined an intuitionistic ordered weighted averaging (OWA) operator to integrate several IFS and presented a simple application of the new intuitionistic OWA operator in multiple-expert multi-criteria decision making. Xu [16] proposed intuitionistic fuzzy ordered weighted averaging (IFOWA) operator and intuitionistic fuzzy hybrid averaging (IFHA) operator. Xu and Yager [17] developed several geometric aggregation operators, such as intuitionistic fuzzy weighted geometric (IFWG) operator, intuitionistic fuzzy ordered weighted geometric (IFOWG) operator, and intuitionistic fuzzy hybrid geometric (IFHG) operator, and presented an application of the IFHG operator to multi-attribute group decision making (MAGDM) with intuitionistic fuzzy information.

Due to the motivation and inspiration of the above discussion, we generalized the concept of Pythagorean trapezoidal linguistic fuzzy set which give us more accurate and precious result as compare to the above mention operators. Thus keeping an advantage of the above mention aggregation operators. In this paper, we develop a series of cubic aggregation operators such that cubic weighted averaging (CWA) operator, cubic ordered weighted averaging (COWA) operator and cubic hybrid averaging (CHA) operator.

The rest of this paper is structured as follows. In section 2, we review some basic Definition and operational laws which are used our latter work. In section 3, we analyze different types of cubic weighted averaging (CWA) operator, cubic ordered weighted averaging (COWA) operator and cubic hybrid averaging (CHA) operator and also study its various properties [18-21]. In section 4 briefly describe the decision making process based on developed operators and we give a numerical example in section 5. In section 6 summarizes the main conclusions of the paper.

Preliminaries

Definition 1. [2] Let a set X be fixed. An (IFS) A in X is an object having the form:

\[ A = \{ (x, \mu_A(x), \nu_A(x)) | x \in X \} \]  

where the functions \( \mu_A : X \rightarrow [0,1] \) and \( \nu_A : X \rightarrow [0,1] \) define the degree of membership and the degree of non-membership of the element \( x \in X \) to A respectively, and for every

\[ x \in X : 0 \leq \mu_A(x) + \nu_A(x) \leq 1 \]  

For each (IFS) A in X, if

\[ \pi_A(x) = 1 - \mu_A(x) - \nu_A(x), \text{ for all } x \in X \]

then \( \pi_A(x) \) is called the degree of indeterminacy of \( x \) to \( A \).

Definition 2. [9] Let \( X \) be a fixed non empty set. A cubic set is an object of the form,

\[ C = \{ (a, A(a), \lambda(a)) : a \in X \} , \]

where \( A \) is an (IVFS) and \( \lambda \) is a fuzzy set in \( X \). A cubic set \( C = \{ (a, A(a), \lambda(a)) \} \) is simply denoted by \( \tilde{C} = \{ \tilde{A}, \tilde{\lambda} \} = \{ [a, a^*], \lambda \} \). The collection of all cubic set is denoted by \( C(X) \).

(a) if \( \lambda \in \tilde{A}(x) \) for all \( x \in X \) so it is called interval cubic set;
(b) If \( \lambda \notin \tilde{A}(x) \) for all \( x \in X \) so it is called external cubic set;
(c) If \( \lambda \notin \tilde{A}(x) \) or \( \lambda \notin \tilde{A}(x) \) its called cubic set for all \( x \in X \).

Definition 3. [9] Let \( A = \{ \tilde{A}, \lambda \} \) and \( B = \{ \tilde{B}, \mu \} \) be cubic set in \( X \), then we define

(a) (Equality) \( A = B \iff A = B \) and \( \lambda = \mu \).
(b) (P-order) \( A \subseteq B \iff A \subseteq B \) and \( \lambda \leq \mu \).
(c) (R-order) \( A \subseteq B \iff A \subseteq B \) and \( \lambda \leq \mu \).

Definition 4. [9] The complement of \( A = \{ \tilde{A}, \lambda \} \) is defined to be the cubic set

\[ A^c = \{ (x, A^c(x), 1 - \lambda(x)) \} | x \in X \} . \]

Cubic Numbers, Score and Accuracy Function

In this section we define the some operational laws of cubic numbers. We define score function and accuracy function of a cubic set.
Definition 5. Let \( C = \{A, \lambda\}, C_1 = \{A_1, \lambda_1\}, \) and \( C_2 = \{A_2, \lambda_2\} \) be three (CVs), then the following operational laws holds.

1. \( C_1 \oplus C_2 = \left[\begin{array}{c} \tilde{a}_1 + \tilde{a}_2, \tilde{a}_1 - \tilde{a}_2, a_1^\ast - a_2^\ast, a_1 - a_2, \lambda_1, \lambda_2 \end{array}\right] \triangledown \lambda \)

2. \( C_1 \otimes C_2 = \left[\begin{array}{c} \tilde{a}_1, a_1^\ast, a_1^\ast, a_1^\ast + a_2^\ast, a_1 - a_2, \lambda_1, \lambda_2 \end{array}\right] \triangledown \lambda \)

3. \( \delta C = \left[\begin{array}{c} 1 - (1 - \tilde{a}_1)^\delta, \lambda_1 \end{array}\right] \triangledown \lambda \)

4. \( C^\delta = \left[\begin{array}{c} (a_1^\ast)^\delta, (a_2^\ast)^\delta, 1 - (1 - \lambda_1)^\delta \end{array}\right] \triangledown \lambda \)

Based on the (CVs) we introduce a score function \( S \) to evaluate the degree of suitability that an alternative satisfies a decision maker’s requirement. Let \( C = \{A, \lambda\} \) be an (CV), where \( \tilde{A} \in [0,1], \lambda \in [0,1] \)

The score of \( C \) can be evaluated by the score function \( S \) shown as

\[
s(C) = \frac{\tilde{A} - \lambda}{3} = \frac{\tilde{a} + a^\ast - \lambda}{3} \tag{4}\]

Now an accuracy function to evaluate the degree of accuracy of the (CV) \( C = \{A, \lambda\} \) as

\[
h(C) = \frac{1 + \tilde{A} - \lambda}{3} = \frac{1 + \tilde{a} + a^\ast - \lambda}{3} \tag{5}\]

where \( h(C) \in [0,1] \).

The larger the value of \( h(C) \), the higher the degree of accuracy of the degree of membership of the (CV). Then, utilized the score function, the accuracy function, and the minimum and maximum operations to develop another technique for handling multiple attribute decision-making problems based on cubic information.

Remarks

1. If \( s(C) < s(D) \), then \( C \) is smaller than \( D \), denoted by \( C < D \);
2. If \( s(C) = s(D) \), then we have;
   i. If \( h(C) = h(D) \), then \( C \) and \( D \) represent the same information, denoted by \( C = D \);
   ii. If \( h(C) < h(D) \), then \( C \) is smaller than \( D \), denoted by \( C < D \).

(CWA,COWA and CHA) Operators

Definition 6. Let \( \Omega \) be the set of all cubic values and \( C_j = \{A_j, \lambda_j\} = \{(a_j, a_j^\ast), \lambda_j\} \)

\((j=1,2,...,n)\) be a collection of cubic values, and let \( CWA : \Omega^n \rightarrow \Omega \), if

\[
CWA(c_1, c_2, ..., c_n) = \frac{w_1c_1 \oplus w_2c_2 \oplus ... \oplus w_nc_n}{\sum_{j=1}^{n}w_j} = \left[\begin{array}{c} 1 - \prod_{j=1}^{n}(1 - \tilde{a}_j)^\gamma, 1 - \prod_{j=1}^{n}(1 - a_j^\ast)^\gamma \end{array}\right] \triangledown \lambda \]

where \( w=(w_1, w_2, ..., w_n)^T \) is weight vector of \( C_j \) \((j=1,2,...,n)\) with \( w_j \in [0,1] \) and \( \sum_{j=1}^{n}w_j = 1 \).

Example 1. Let \( C_1 = \{(0.3, 0.6, 0.6), (0.6, 0.7, 0.4), (0.1, 0.5, 0.3) \) and \( C_4 = \{(0.2, 0.3, 0.5) \) be four cubic values, and suppose that \( w=(0.30, 0.20, 0.40, 0.10)^T \) be the weight vector of \( C_j \) \((j=1,2,...,4)\). Using equation (6) such that
Let \( COWA \) operator of dimension \( n \) is a mapping \( COWA : \Omega^n \rightarrow \Omega \), that has an associated vector \( w=(w_1, w_2, \ldots, w_n)^T \) is weight vector of \( C_j \) \((j=1,2,\ldots, n)\) with \( w_j \in [0,1] \) and \( \sum_{j=1}^{n} w_j = 1 \), such that

\[
COWA(c_1, c_2, \ldots, c_n) = \left[ 1 - \prod_{j=1}^{n} (1 - a_{c_j})^{v_j}, 1 - \prod_{j=1}^{n} (1 - a_{c_j})^{v_j} \right],
\]

where \( c_{(j)} \) is the \( j \)th largest of the \( c_j \). Especially, if \( w = \left( \frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n} \right)^T \), then the \( COWA \) operator is reduced to the CA operator.

Example 2. Let \( C_1 = \{0.3,0.5,0.2\}, C_2 = \{0.2,0.4,0.1\}, C_3 = \{0.6,0.7,0.5\}, C_4 = \{0.4,0.8,0.5\} \) and \( C_5 = \{0.1,0.3,0.2\} \) be five cubic values, and suppose that \( w = (0.25,0.15,0.10,0.20,0.30)^T \) be the weight vector of \( C_j \) \((j=1,2,\ldots,5)\). First of all we find the score function by applying Eq (4) such that

\[
S(c_1) = \frac{0.3 + 0.5 - 0.2}{3} = 0.20, \quad S(c_2) = \frac{0.2 + 0.4 - 0.1}{3} = 0.16
\]

\[
S(c_3) = \frac{0.6 + 0.7 - 0.5}{3} = 0.26, \quad S(c_4) = \frac{0.4 + 0.8 - 0.5}{3} = 0.23
\]

\[
S(c_5) = \frac{0.1 + 0.3 - 0.3}{3} = 0.03.
\]

Since \( S(c_3) > S(c_2) > S(c_4) > S(c_5) \) Hence

\[
C_{\sigma(1)} = \{0.6,0.7,0.5\}, C_{\sigma(2)} = \{0.4,0.8,0.5\}, \]

\[
C_{\sigma(3)} = \{0.3,0.5,0.2\}, C_{\sigma(4)} = \{0.2,0.4,0.1\}, \]

\[
C_{\sigma(5)} = \{0.1,0.3,0.2\}.
\]

Applying Equation (7) and \( w = \left(0.25,0.15,0.10,0.20,0.30\right)^T \) is the weighting vector the \( COWA \) operator such that

\[
COWA(c_1, c_2, \ldots, c_n) = \left[ 1 - \prod_{j=1}^{n} (1 - a_{c_j})^{v_j}, 1 - \prod_{j=1}^{n} (1 - a_{c_j})^{v_j} \right],
\]

\[
= \left[ 0.29, 0.56 \right], 0.41.
\]

The fundamental aspect of the \( COWA \) operator is the reordering of the arguments to be aggregated, based on their values. The \( COWA \) operator has different properties which we discuss as follows

**Theorem 1.** (Commutativity)

\[
COWA_w(c_1, c_2, \ldots, c_n) = COWA_w(c_1', c_2', \ldots, c_n')
\]

where \((c_1', c_2', \ldots, c_n')\) is any permutation of \((c_1, c_2, \ldots, c_n)\).
Proof. Since

\[ COWA_n(c_1, c_2, \ldots, c_n) = w_i c_{a_i} \oplus w_2 c_{a_2} \oplus \ldots \oplus w_n c_{a_n}, \]

\[ COWA_n(c_1', c_2', \ldots, c_n') = w_i c_{a_i} \oplus w_2 c_{a_2} \oplus \ldots \oplus w_n c_{a_n}, \]

since \((c_1', c_2', \ldots, c_n')\) is any permutation of \((c_1, c_2, \ldots, c_n)\) then

\[ COWA_n(c_1, c_2, \ldots, c_n) = COWA_n(c_1', c_2', \ldots, c_n'). \]

**Theorem 2. (Idempotency)**

If \(C_j = C \forall j\) for all \(j = 1, 2, \ldots, n\) where \(C = (\lambda_1, \lambda_2, \ldots, \lambda_n)\), then

\[ COWA_n(c_1, c_2, \ldots, c_n) = c. \]

Proof. Since \(C_j = C \forall j\), we have

\[ COWA_n(c_1, c_2, \ldots, c_n) = w_i c_{a_i} \oplus w_2 c_{a_2} \oplus \ldots \oplus w_n c_{a_n}, \]

\[ = w_i c \oplus w_2 c \oplus \ldots \oplus w_n c \]

\[ = \left[ 1 - \prod_{j=1}^{n}(1 - \lambda_j)^{w_j}, 1 - \prod_{j=1}^{n}(1 - \lambda_j)^{w_j} \right], \]

\[ = \left[ 1 - \prod_{j=1}^{n}(\lambda_j)^{w_j}, 1 - \prod_{j=1}^{n}(\lambda_j)^{w_j} \right], \]

since \(\sum_{j=1}^{n} w_j = 1\) then we can get

\[ COWA_n(c_1, c_2, \ldots, c_n) = \left[ (\lambda_1, \lambda_2, \ldots, \lambda_n) \right] = C. \]

**Theorem 3. (Monotonicity)**

If \(C_j \leq C_j\) for all \(j = 1, 2, \ldots, n\)

\[ COWA_n(c_1, c_2, \ldots, c_n) \leq COWA_n(c_1', c_2', \ldots, c_n'). \]

Proof. Let

\[ COWA_n(c_1, c_2, \ldots, c_n) = w_i c_{a_i} \oplus w_2 c_{a_2} \oplus \ldots \oplus w_n c_{a_n}, \]

\[ COWA_n(c_1', c_2', \ldots, c_n') = w_i c_{a_i} \oplus w_2 c_{a_2} \oplus \ldots \oplus w_n c_{a_n}. \]

Since \(C_j \leq C_j\) for all \(j\) it follows that \(C_{a(j)} \leq C_{a(j)}\) where \(j = 1, 2, \ldots, n\) then

\[ COWA_n(c_1, c_2, \ldots, c_n) \leq COWA_n(c_1', c_2', \ldots, c_n'). \]

The **CHA Operator**

Consider that the CWA operator weights only the cubic value set, whereas the COWA operator weights only the ordered positions of the CVs instead of the weighting the cubic value set themselves. To overcome this limitation, motivated by the idea of combining the WA and OWA operators, in what follows, we developed a cubic hybrid averaging aggregation (CHA) operator, which weights both the given cubic value and its ordered position.

**Definition 8.** CHA operator of dimension \(n\) is a mapping \(CHA : \Omega^n \rightarrow \Omega\), which has an associated vector \(w = (w_1, w_2, \ldots, w_n)^T\), with \(w_j \geq 0, j = 1, 2, \ldots, n\), and \(\sum_{j=1}^{n} w_j = 1\), such that
where $c_{nj}$ is the $j$th largest of the weighted CV's $c_j = mc_j$ for $j = 1, 2, \ldots, n$ with $w_j \geq 0$, and $n$ is balancing coefficient, which plays a role of balance if the vector $w = (w_1, w_2, ..., w_n)^T$ approaches $(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})^T$, then the vector $(nw_1 c_1, nw_2 c_2, ..., nw_n c_n)^T$ approaches $(c_1, c_2, ..., c_n)^T$. Let $C_{nj(i)} = \left( A_{nj(i)}, \lambda_{nj(i)} \right)$. Then, similar to Theorem 3, we have

$$\text{CHA}(c_1, \ldots, c_n) = w_1 c_{nj(i)} \oplus w_2 c_{nj(i)} \oplus \ldots \oplus w_n c_{nj(i)}$$

$$= \left[ 1 - \prod_{j=1}^{n} (1 - \lambda_{nj(i)})^{w_j}, 1 - \prod_{j=1}^{n} (1 - \lambda_{nj(i)})^{w_j} \right]$$

which is called cubic hybrid averaging (CHA operator).

**Example 3.** Let $C_1 = \left\{ [0.3, 0.4], 0.3 \right\}$, $C_2 = \left\{ [0.5, 0.6], 0.7 \right\}$, $C_3 = \left\{ [0.4, 0.3], 0.5 \right\}$, $C_4 = \left\{ [0.6, 0.8], 0.3 \right\}$ and be four cubic values, and suppose that $w = (0.20, 0.25, 0.15, 0.40)^T$ be the weight vector of $C_j$ $(j = 1, 2, \ldots, 4)$, then by operational law (3) in Definition (5), we get the weighted cubic values such that

\[
\begin{align*}
    c_1 &= \left\{ [0.24, 0.33], 0.38 \right\}, \quad c_2 = \left\{ [0.50, 0.60], 0.70 \right\} \\
    c_3 &= \left\{ [0.26, 0.19], 0.65 \right\}, \quad c_4 = \left\{ [0.76, 0.92], 0.14 \right\}
\end{align*}
\]

By applying equation (4), we calculate the scores of $C_j$ $(j = 1, 2, \ldots, 4)$:

\[
\begin{align*}
    S(c_1) &= \frac{0.24 + 0.33 - 0.38}{3} = 0.060, \\
    S(c_2) &= \frac{0.50 + 0.60 - 0.70}{3} = 0.1333, \\
    S(c_3) &= \frac{0.26 + 0.19 - 0.65}{3} = -0.0666, \\
    S(c_4) &= \frac{0.76 + 0.92 - 0.14}{3} = 0.5133
\end{align*}
\]

Since $S(c_1) \geq S(c_2) \geq S(c_3) \geq S(c_4)$, hence we can write as follows

\[
\begin{align*}
    c_{m(1)} &= \left\{ [0.76, 0.92], 0.14 \right\}, \quad c_{m(2)} = \left\{ [0.50, 0.60], 0.70 \right\} \\
    c_{m(3)} &= \left\{ [0.24, 0.33], 0.38 \right\}, \quad c_{m(4)} = \left\{ [0.26, 0.19], 0.65 \right\}
\end{align*}
\]

If the associated weighting vector is $\omega = (0.30, 0.10, 0.35, 0.15)^T$, then we aggregate values by using the equation (9) such that

\[
\begin{align*}
    \text{CHA}_{\omega}(c_1, c_2, ..., c_4) &= \left\{ 1 - \prod_{j=1}^{n} (1 - \lambda_{nj(i)})^{w_j}, 1 - \prod_{j=1}^{n} (1 - \lambda_{nj(i)})^{w_j} \right\} \\
    &= \left\{ \prod_{j=1}^{n} (\lambda_{nj(i)})^{w_j}, \prod_{j=1}^{n} (\lambda_{nj(i)})^{w_j} \right\} \\
    &= \left\{ [0.4799, 0.6580], [0.3687] \right\}
\end{align*}
\]

**Theorem 4.** The CWA operator is a special case of the CHA operator.
then is an attribute value provided by decision maker’s Dq, denoted by a cubic numbers \((c_{\alpha ij})\). and consider, where \(T_i\) into the collective cubic decision matrix respectively so we have \(X\) be an cubic decision matrix, where \(\lambda\) so we have be an cubic decision matrix, where \(\lambda\) and want to select best one alternative. There are four experts \(D_q\) for this purpose let \((x_i, x_{i1}, x_{i2}, x_{i3})\) are four alternatives such that \(X_i(\alpha = 1, 2, 3, 4)\) and \(\lambda\) such that \(\lambda = (0.20, 0.30, 0.15, 0.35)\). Consider there are four attributes \((a_{\alpha ij})\) such that \(U_j = (j = 1, 2, 3, 4)\).\n
\[\text{Step 7:} \quad \text{End.}\]

\[\text{Step 6:} \quad \text{In this step, we arrange the score values of each alternative by descending order and chose the best alternative by maximum value of score function.}\]

\[\text{Step 5:} \quad \text{In this step, we compute the scores of } r_i (i = 1, 2, ..., m). \text{If there is difference between two or more than two score functions then we have must to calculate the accuracy degrees.}\]

\[\text{Step 4:} \quad \text{Aggregate all the preference values } r_i = \{\lambda_{ij}\} \text{ where } (i = 1, 2, ..., m) \text{ and } (j = 1, 2, ..., n) \text{ by using COWA operator to get the complete preference value } r_i \text{ corresponding to the alternative } X_i (i = 1, 2, ..., m)\]

\[\text{Step 3:} \quad \text{Applying the COWA operators of equation (7) to aggregate all the individual cubic decision matrix}\]

\[R_k = (r_{ij}^{(k)})_{4x4} (k = 1, 2, 3, 4) \text{ into the collective cubic decision matrix } R = (r_{ij})_{4x4}\]

\[\text{Step 2:} \quad \text{In this step, we apply the score function and construct the ordered decision matrices.}\]

\[\text{Step 1:} \quad \text{In this step, first of all we Construct the individual cubic decision matrix.}\]

\[\text{Algorithm}\]

\[GCHA_{n, w}(c_{11}, c_{12}, ..., c_{n1}) = (w_1(c_{11}) \oplus w_2(c_{12}) \oplus ... \oplus w_n(c_{n1})) = \frac{1}{n}((c_{\alpha ij}) \oplus (c_{\alpha ij}) \oplus ... \oplus (c_{\alpha ij})) = (w(c_{\alpha ij}) \oplus w(c_{\alpha ij}) \oplus ... \oplus w(c_{\alpha ij})) = CWA_{n}(c_{11}, c_{12}, ..., c_{n1}).\]

which completes the proof of Theorem.

**Approach to Multiple Attribute Group Decision Making with Cubic Information**

In this section, we apply the aggregation operator developed to multiple attribute group decision making with cubic information: For this purpose we consider \(X = \{x_i, x_{i1}, x_{i2}, x_{i3}\}\) be a set of \(m\) alternatives, \(U = \{u_i, u_{i1}, u_{i2}\}\) be a set of \(n\) attributes, having weighting vector is \(w = (w_1, w_2, ..., w_n)^T\), \(w_j \geq 0, j = 1, 2, ..., n\), and \(\sum_{j=1}^n w_j = 1\) and consider \(D = \{d_1, d_{i1}, d_{i2}\}\) be set of \(q\) decision makers, whose weighting vector is \(\lambda = \{\lambda_1, \lambda_2, ..., \lambda_q\}^T\), where \(\lambda_i \in [0, 1], i = 1, 2, ..., q\), and \(\sum_{j=1}^q \lambda_j = 1\). Let \(A = (a_{ij})_{4x4}\) be an cubic decision matrix, where \(a_{ij}^{(k)} = \{\lambda_{ij}^{(k)}\} \text{ is an attribute value provided by decision maker’s Dq, denoted by a cubic numbers such that } \lambda_{ij}^{(k)} \in [0, 1], \lambda_{ij}^{(k)} \in [0, 1] \text{ i = 1, 2, ..., m; j = 1, 2, ..., n}\.

Then we apply the COWA operator to develop an approach to multiple attribute group decision making with cubic information, which consist of the following steps.

\[\text{Algorithm}\]

\[\text{Step 1:} \quad \text{In this step, first of all we Construct the individual cubic decision matrix.}\]

\[\text{Step 2:} \quad \text{In this step, we apply the score function and construct the ordered decision matrices.}\]

**Theorem 5.** The COWA operator is a special case of the CHA operator.

**Proof.** Let \(w = (\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})^T\), then \(C_i = C_i (i = 1, 2, ..., n)\), so we have

\[GCHA_{n, w}(c_{11}, c_{12}, ..., c_{n1}) = (w_1(c_{11}) \oplus w_2(c_{12}) \oplus ... \oplus w_n(c_{n1})) = \frac{1}{n}((c_{\alpha ij}) \oplus (c_{\alpha ij}) \oplus ... \oplus (c_{\alpha ij})) = (w(c_{\alpha ij}) \oplus w(c_{\alpha ij}) \oplus ... \oplus w(c_{\alpha ij})) = CWA_{n}(c_{11}, c_{12}, ..., c_{n1}).\]
• $u_1$ is the language of a thesis;
• $u_2$ is the innovation;
• $u_3$ is the rigor;
• $u_4$ is the structure of the thesis.

Where $w=(0.1,0.2,0.3,0.4)^T$ is weighting vector of attributes $U_j=(j=1,2,3,4)$. The experts $D_q \ (q=1,2,3,4)$ evaluate the thesis $X_i \ (i=1,2,3,4)$ with respect to attributes such that $U_j=(j=1,2,3,4)$, and develop four cubic decision matrix such that $R_k=(u^{(k)}_j)_{4 \times 4} \ (k=1,2,3,4)$.

**Step 1:** The decision makers give his decision in the following tables.

**Table 1:** Individual cubic decision matrix $R^{(1)}$

<table>
<thead>
<tr>
<th></th>
<th>$u_1$</th>
<th>$u_2$</th>
<th>$u_3$</th>
<th>$u_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>[0.5,0.7],0.5</td>
<td>[0.4,0.6],0.9</td>
<td>[0.3,0.5],0.2</td>
<td>[0.6,0.8],0.3</td>
</tr>
<tr>
<td>$x_2$</td>
<td>[0.3,0.6],0.7</td>
<td>[0.5,0.7],0.8</td>
<td>[0.5,0.6],0.3</td>
<td>[0.3,0.6],0.7</td>
</tr>
<tr>
<td>$x_3$</td>
<td>[0.7,0.8],0.2</td>
<td>[0.4,0.8],0.2</td>
<td>[0.6,0.8],0.5</td>
<td>[0.5,0.4],0.6</td>
</tr>
<tr>
<td>$x_4$</td>
<td>[0.2,0.7],0.6</td>
<td>[0.5,0.9],0.3</td>
<td>[0.4,0.9],0.1</td>
<td>[0.3,0.4],0.5</td>
</tr>
</tbody>
</table>

**Table 2:** Individual cubic decision matrix $R^{(2)}$

<table>
<thead>
<tr>
<th></th>
<th>$u_1$</th>
<th>$u_2$</th>
<th>$u_3$</th>
<th>$u_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>[0.3,0.4],0.3</td>
<td>[0.3,0.4],0.2</td>
<td>[0.5,0.7],0.6</td>
<td>[0.4,0.6],0.3</td>
</tr>
<tr>
<td>$x_2$</td>
<td>[0.2,0.6],0.4</td>
<td>[0.8,0.9],0.5</td>
<td>[0.5,0.7],0.5</td>
<td>[0.4,0.8],0.1</td>
</tr>
<tr>
<td>$x_3$</td>
<td>[0.6,0.7],0.2</td>
<td>[0.3,0.4],0.4</td>
<td>[0.5,0.7],0.4</td>
<td>[0.3,0.5],0.4</td>
</tr>
<tr>
<td>$x_4$</td>
<td>[0.6,0.8],0.1</td>
<td>[0.4,0.5],0.7</td>
<td>[0.5,0.9],0.3</td>
<td>[0.3,0.4],0.2</td>
</tr>
</tbody>
</table>

**Table 3:** Individual cubic decision matrix $R^{(3)}$

<table>
<thead>
<tr>
<th></th>
<th>$u_1$</th>
<th>$u_2$</th>
<th>$u_3$</th>
<th>$u_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>[0.2,0.5],0.3</td>
<td>[0.3,0.6],0.1</td>
<td>[0.3,0.4],0.3</td>
<td>[0.4,0.5],0.3</td>
</tr>
<tr>
<td>$x_2$</td>
<td>[0.1,0.4],0.1</td>
<td>[0.3,0.8],0.8</td>
<td>[0.2,0.3],0.3</td>
<td>[0.2,0.5],0.6</td>
</tr>
<tr>
<td>$x_3$</td>
<td>[0.4,0.7],0.3</td>
<td>[0.5,0.7],0.6</td>
<td>[0.4,0.9],0.3</td>
<td>[0.3,0.5],0.3</td>
</tr>
<tr>
<td>$x_4$</td>
<td>[0.3,0.5],0.3</td>
<td>[0.3,0.5],0.5</td>
<td>[0.3,0.7],0.4</td>
<td>[0.4,0.5],0.2</td>
</tr>
</tbody>
</table>

**Table 4:** Individual cubic decision matrix $R^{(4)}$

<table>
<thead>
<tr>
<th></th>
<th>$u_1$</th>
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<th>$u_3$</th>
<th>$u_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>[0.4,0.6],0.9</td>
<td>[0.2,0.5],0.5</td>
<td>[0.2,0.5],0.4</td>
<td>[0.4,0.5],0.3</td>
</tr>
<tr>
<td>$x_2$</td>
<td>[0.5,0.8],0.3</td>
<td>[0.2,0.4],0.5</td>
<td>[0.4,0.5],0.7</td>
<td>[0.3,0.6],0.2</td>
</tr>
<tr>
<td>$x_3$</td>
<td>[0.2,0.3],0.4</td>
<td>[0.1,0.5],0.5</td>
<td>[0.3,0.9],0.5</td>
<td>[0.3,0.5],0.3</td>
</tr>
<tr>
<td>$x_4$</td>
<td>[0.5,0.5],0.3</td>
<td>[0.3,0.5],0.5</td>
<td>[0.3,0.7],0.4</td>
<td>[0.4,0.5],0.5</td>
</tr>
</tbody>
</table>

**Step 2:** In this step we applying equation (4) and find the ordered of the ordered of Individual cubic decision matrix.

**Table 5:** Individual cubic ordered decision matrix $R^{(1)}$

<table>
<thead>
<tr>
<th></th>
<th>$u_1$</th>
<th>$u_2$</th>
<th>$u_3$</th>
<th>$u_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>[0.6,0.8],0.3</td>
<td>[0.5,0.7],0.5</td>
<td>[0.3,0.5],0.2</td>
<td>[0.4,0.6],0.9</td>
</tr>
<tr>
<td>$x_2$</td>
<td>[0.5,0.6],0.3</td>
<td>[0.5,0.7],0.8</td>
<td>[0.3,0.6],0.7</td>
<td>[0.3,0.6],0.7</td>
</tr>
<tr>
<td>$x_3$</td>
<td>[0.7,0.8],0.2</td>
<td>[0.4,0.8],0.2</td>
<td>[0.6,0.8],0.5</td>
<td>[0.5,0.4],0.6</td>
</tr>
<tr>
<td>$x_4$</td>
<td>[0.4,0.9],0.1</td>
<td>[0.5,0.9],0.3</td>
<td>[0.2,0.7],0.6</td>
<td>[0.4,0.9],0.1</td>
</tr>
</tbody>
</table>
Step 3: Using the COWA operator to aggregate all the different cubic ordered decision matrices, into the single matrix.

\[
R^{(2)} = \begin{bmatrix}
    x_1 & [0.4,0.6],0.3 & [0.5,0.7],0.6 & [0.3,0.4],0.2 & [0.3,0.4],0.3 \\
    x_2 & [0.8,0.9],0.5 & [0.4,0.8],0.1 & [0.5,0.7],0.5 & [0.2,0.6],0.4 \\
    x_3 & [0.6,0.7],0.2 & [0.5,0.7],0.4 & [0.3,0.5],0.4 & [0.3,0.4],0.4 \\
    x_4 & [0.6,0.8],0.1 & [0.3,0.4],0.2 & [0.1,0.4],0.4 & [0.3,0.4],0.2 \\
\end{bmatrix}
\]

Table 6: Individual cubic ordered decision matrix \(R^{(2)}\)

\[
R^{(3)} = \begin{bmatrix}
    x_1 & [0.3,0.6],0.1 & [0.4,0.5],0.3 & [0.2,0.5],0.3 & [0.3,0.4],0.3 \\
    x_2 & [0.1,0.4],0.1 & [0.3,0.8],0.8 & [0.2,0.3],0.3 & [0.2,0.5],0.6 \\
    x_3 & [0.4,0.9],0.3 & [0.4,0.7],0.3 & [0.5,0.7],0.6 & [0.3,0.5],0.3 \\
    x_4 & [0.4,0.5],0.2 & [0.3,0.7],0.4 & [0.3,0.5],0.3 & [0.3,0.5],0.5 \\
\end{bmatrix}
\]

Table 7: Individual cubic ordered decision matrix \(R^{(3)}\)

\[
R^{(4)} = \begin{bmatrix}
    x_1 & [0.4,0.6],0.9 & [0.4,0.5],0.3 & [0.2,0.5],0.4 & [0.2,0.5],0.5 \\
    x_2 & [0.5,0.8],0.3 & [0.3,0.6],0.2 & [0.4,0.5],0.7 & [0.2,0.4],0.5 \\
    x_3 & [0.3,0.9],0.5 & [0.3,0.5],0.3 & [0.1,0.5],0.2 & [0.2,0.3],0.4 \\
    x_4 & [0.5,0.5],0.3 & [0.3,0.7],0.4 & [0.4,0.5],0.5 & [0.3,0.5],0.5 \\
\end{bmatrix}
\]

Table 8: Individual cubic ordered decision matrix \(R^{(4)}\)

Step 4: Now in this step we aggregate all the preference values \(r_{ij} = (A_{ij}, \lambda_{ij})\) where \((i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)\) by using COWA operator to get the complete preference value \(r_i\) corresponding to the alternative \(X_i (i=1,2,\ldots,m)\)

\[
(\lambda) = ([0.3408,0.5366],[0.3584]),(\lambda) = ([0.3975,0.5946],[0.3723])
\]

\[
(\lambda) = ([0.3546,0.6358],[0.3378]),(\lambda) = ([0.3480,0.6035],[0.3116])
\]

Step 5: In this step we compute the scores of \(r_i (i=1,2,3,4)\)

\[
S(r_1) = 0.1730, S(r_2) = 0.2066, S(r_3) = 0.2175, S(r_4) = 0.2133.
\]

Step 6: Now we arrange the score values of each alternative by descending order and chose the best alternative by maximum value of score function \(S(r_1) > S(r_2) > S(r_3) > S(r_4)\). Thus \(x_3 > x_4 > x_2 > x_1\); hence \(x_3\) is best alternative.

Step 7: End.

Further Discussion

In order to show the validity and effectiveness of the proposed methods, we utilize intuitionisti fuzzy (IFS) sets to solve the same problem described above. We apply the proposed aggregation operators developed in this paper. After simplification we can get the ranking result as \(x_3 > x_4 > x_2 > x_1\), we find that \(x_3\) is best alternative. In the above example, if we use IFS sets to express the decision maker’s evaluations then the decision matrix \(R^{(1)}, R^{(2)}, R^{(3)}, R^{(4)}\) can be written as decision matrix \(R^{(11)}, R^{(22)}, R^{(33)}, R^{(44)}\) deleting the cubic numbers which are shown respectively. In [20], the proposed IFWA operators to deal with multiple attribute decision making with intuitionistic fuzzy information respectively such that,
We further explain to find the best alternative of IFS, after the computation process of the overall preference values are as follows.

By applying score function of such that,

\[ S(R_1) = -0.0176, \quad S(R_2) = 0.0252, \quad S(R_3) = 0.0168, \quad S(R_4) = 0.0364. \]

Now we find the ranking as \( x_4 > x_2 > x_3 > x_1 \). In this case, \( x_4 \) is the best alternative. It is noted that the ranking orders obtained by this paper and by [20] are very different. Therefore, CFNs may better reflect the decision information than IFNs. Hence our proposed approach is better than IFNs.

**Conclusion**

In this paper, we have investigated the multiple attribute group decision making problems in which attribute values are in the form of cubic numbers. We first define some operations of CNs and some corresponding operational laws. Further, we have proposed some new aggregation operators for CNs, including cubic weighted averaging (CWA) operator, cubic order weighted averaging (COWA) operator and cubic hybrid averaging (CHA) operator. And desirable properties of the operators have also been analyzed. Finally, an illustrative example has been constructed to show the proposed MAGDB method. Our proposed method is different from all the previous techniques for group decision making due to the fact that the proposed method use cubic fuzzy information, which will not cause any loss of information in the process. So it efficient and feasible for real-world decision making applications.

**References**

1. Zadeh LA (1965) This Week's Citation Classic. Fuzzy Sets Inf Control 8: 338-53.


